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The Application of Signal Detection Theory to Optics

PROGRESS REPORT

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ABSTRACT

Research undertaken during the quarter ending December 15, 1969 is outlined. Topics included estimation of parameters of incoherently radiating objects, detection of optical signals of known phase, and computation of coefficients for discrete-Fourier-transform filters.

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Summary

Research has continued on quantum detection and estimation theory and on filtering by means of the discrete Fourier transform. Quantum estimation theory has been concerned with the estimation of parameters of incoherently radiating objects, such as the location and brightness of a star or the mapping of the radiance distribution of the object plane of an optical instrument. In this study it is convenient to use the threshold approximation, which replaces the logarithm of the likelihood functional by the first term of its expansion in powers of the signal-to-noise ratio. The validity of this approximation has been investigated, and the results are described in Section I of this report.

The detectability of coherent optical signals as in a binary communication system maintaining phase control depends on the solution of a certain eigenvalue problem. Analytical methods having thus far failed, a numerical solution has been programmed for the digital computer. Results show a continuous transition from the known exact solution for zero thermal noise to the asymptotic, quasi-classical solution. Details are given in Section 2.

Further work on the specification of discrete-Fourier-transform filters is reported in Section 3. A method for calculating coefficients of the causal filter minimizing the mean-square estimation error has been developed. Application to image restoration is contemplated.

In addition, research in radiance mapping of an object plane is being pursued with the aim of exploiting the analogy between it and the estimation of the power spectral density of a temporal stochastic process by observation of a filtered version of the process itself. The noise characteristics of image amplifying and detecting devices are being studied to provide statistical information needed for applying the methods of detection and estimation theory.

I. Validity of the Threshold Approximation for Ambiguity Functions

The ambiguity function for optical parameter estimation as defined in previous papers is[1,2]

$$H_0(\theta_1, \theta_2) = N^{-2} \int_A \int_0^T \int_A \int_0^T \varphi_S(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2; \theta_1)$$

$$\times \varphi_S(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1; \theta_2)\, d^2\mathbf{r}_1 d^2\mathbf{r}_2 dt_1 dt_2, \tag{1}$$

where $\varphi_S(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2; \theta)$ is the spatio-temporal autocovariance function of the light from the object as observed at the aperture A of the optical instrument during $(0, T)$, θ is a parameter of the object such as its radiance or its position and is to be estimated, and N is the spectral density of the white background radiation. The Cramér-Rao bound on the mean-square error of an estimate of θ is expressed in terms of the derivative $\partial^2 H(\theta_1, \theta_2)/\partial\theta_1\partial\theta_2$ evaluated for the true value of the parameter, $\theta_1 = \theta_2 = \theta$.

This ambiguity function is derived by the threshold approximation, and it is necessary to determine the conditions under which the approximation is valid. This can be done by estimating the relative size of the error. The exact form of the ambiguity function is, in matrix form,[3]

$$H(\theta_1, \theta_2) = \mathrm{Tr}\{[\boldsymbol{\varphi}_1(\theta)]^{-1} \boldsymbol{\varphi}_S(\theta_1)\ [\boldsymbol{\varphi}_1(\theta)]^{-1} \boldsymbol{\varphi}_S(\theta_2)\}, \tag{2}$$

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1. C. W. Helstrom, "Detection and Resolution of Incoherent Objects by a Background-Limited Optical System", J. Opt. Soc. Am. 59, 164-175 (Feb., 1969).
2. C. W. Helstrom, "Resolvability of Objects from the Standpoint of Statistical Parameter Estimation", submitted to J. Opt. Soc. Am.
3. See ref. 1, Appendix B, Eq. B4, p. 174.

where $\underset{\sim}{\varphi}_1(\theta)$ is the matrix of coefficients of an expansion of the total autocovariance function

$$\varphi_1(\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2; \theta) = \varphi_s(\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2; \theta) + N\delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\, \delta(t_1 - t_2) \tag{3}$$

of the aperture field in a series of arbitrary functions orthonormal over the aperture A of the observing system and over the observation interval (0, T). Thus

$$\underset{\sim}{\varphi}_1(\theta) = \underset{\sim}{\varphi}_s(\theta) + N\, \underset{\sim}{I}, \tag{4}$$

where $\underset{\sim}{\varphi}_s(\theta)$ is the corresponding covariance matrix for the object field and $\underset{\sim}{I}$ is the identity matrix. In Eq. (2) "Tr" stands for the trace of the matrix product following it.

The threshold expansion of $[\underset{\sim}{\varphi}_1(\theta)]^{-1}$ is, from (4),

$$[\underset{\sim}{\varphi}_1(\theta)]^{-1} = N^{-1}(\underset{\sim}{I} + N^{-1}\underset{\sim}{\varphi}_s)^{-1} = N^{-1} - N^{-2}\underset{\sim}{\varphi}_s + N^{-3}\underset{\sim}{\varphi}_s^{\,2} - + \ldots,$$

which when substituted into (2) yields

$$H(\theta_1, \theta_2) = N^{-2}\, \mathrm{Tr}[\underset{\sim}{\varphi}_s(\theta_1)\, \underset{\sim}{\varphi}_s(\theta_2)] - 2N^{-3}\, \mathcal{R}\ell\, \mathrm{Tr}[\underset{\sim}{\varphi}_s(\theta)\, \underset{\sim}{\varphi}_s(\theta_1)\, \underset{\sim}{\varphi}_s(\theta_2)] + \ldots \tag{5}$$

where "$\mathcal{R}\ell$" stands for the real part of what follows it. Expressing Eq. (5) in terms of the autocovariance function itself, we obtain

$$H(\theta_1, \theta_2) = H_0(\theta_1, \theta_2) - 2N^{-3}\, \mathcal{R}\ell \int_A \int_0^T \int_A \int_0^T \int_A \int_0^T \varphi_s(\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2; \theta) \times \varphi_s(\underset{\sim}{r}_2, t_2; \underset{\sim}{r}_3, t_3; \theta_1)\, \varphi_s(\underset{\sim}{r}_3, t_3; \underset{\sim}{r}_1, t_1; \theta_2) \times d^2\underset{\sim}{r}_1 dt_1 d^2\underset{\sim}{r}_2 dt_2 d^2\underset{\sim}{r}_3 dt_3 + \ldots \tag{6}$$

the first term being the threshold approximation in Eq. (1). It is the second term whose relative size we must estimate.

The autocovariance function φ_s is written[4] as the product of a spatial part $\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta)$ and a temporal part $\chi(\tau)$,

$$\varphi_s(\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2; \theta) = \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta)\ \chi(t_1 - t_2). \tag{7}$$

The latter is normalized so that $\chi(0) = 1$; it is the Fourier transform of the spectral density $X(\omega)$ of the object light, which we take as real;

$$\chi(\tau) = \int_{-\infty}^{\infty} X(\omega)\ e^{-i\omega\tau}\ d\omega/2\pi. \tag{8}$$

The width W of $X(\omega)$ is much greater than T^{-1} for normal observation times; $WT >> 1$. This width is defined by $W = W_2$, where

$$W_n = \left\{ \int_{-\infty}^{\infty} [X(\omega)]^n\ d\omega/2\pi \right\}^{-1/(n-1)}, \quad n > 1, \tag{9}$$

is a generalized bandwidth for the object light. For most spectra W_n has the same order of magnitude as W.

Writing the expansion in Eq. (6) as

$$H(\theta_1, \theta_2) = H_0(\theta_1, \theta_2) - H_1(\theta_1, \theta_2) + \ldots, \tag{10}$$

we get for the second term, by Eqs. (7) and (9),

$$H_1(\theta_1, \theta_2) = 2N^{-3}\, R\ell \int_0^T\int_0^T\int_0^T \chi(t_1 - t_2)\ \chi(t_2 - t_3)\ \chi(t_3 - t_1)\ dt_1 dt_2 dt_3$$

4. C. W. Helstrom, "Detection and Resolution of Incoherent Objects by a Background-Limited Optical System", J. Opt. Soc. Am. 59, 164-175 (February, 1969).

$$\times \int_A \int_A \int_A \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta)\ \varphi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_3; \theta_1)\ \varphi_s(\underset{\sim}{r}_3, \underset{\sim}{r}_1; \theta_2)\ d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 d^2\underset{\sim}{r}_3$$

$$= 2N^{-3}\, T\, W_3^{-2}$$

$$\times \mathrm{R}\ell \int_A \int_A \int_A \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta)\ \varphi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_3; \theta_1)\ \varphi_s(\underset{\sim}{r}_3, \underset{\sim}{r}_1; \theta_2)\ d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 d^2\underset{\sim}{r}_3; \quad (11)$$

here we have used the assumption $WT \gg 1$ to replace two of the integrations over $(0, T)$ by integrations over $(-\infty, \infty)$.

In terms of the radiance distribution $B(\underset{\sim}{u}; \theta)$ of the object, which is at a distance R from the aperture, the spatial autocovariance function $\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta)$ is[5]

$$\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta) = (4\pi R^2)^{-1} \int_0 B(\underset{\sim}{u}; \theta)\ \mathcal{E}(\underset{\sim}{r}_1, \underset{\sim}{u})\ \mathcal{E}^*(\underset{\sim}{r}_2, \underset{\sim}{u})\ d^2\underset{\sim}{u}, \quad (12)$$

where the integration is taken over the object plane 0, and

$$\mathcal{E}(\underset{\sim}{r}, \underset{\sim}{u}) = \exp\left(\frac{ik}{2R}\,|\underset{\sim}{r} - \underset{\sim}{u}|^2\right),$$

$$k = 2\pi/\lambda = \omega/c, \quad (13)$$

λ being the wavelength of the light from the object. The total energy E_s received from the object is

$$E_s = \int_A \int_0^T \varphi_s(\underset{\sim}{r}, \underset{\sim}{r}\ ; \theta)\ d^2\underset{\sim}{r}\,dt = B_T\ AT/4\pi R^2, \quad (14)$$

with

$$B_T = \int_0 B(\underset{\sim}{u})\ d^2\underset{\sim}{u} \quad (15)$$

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5. See ref. 1, Eqs. (1.8), (1.9), p. 166.

the total object radiance. When Eq. (12) is substituted into Eq. (11), we get after some calculation

$$H_1(\theta_1,\theta_2) = 2(E_S/N)^3 \; (TW_3)^{-2} \; B_T^{-3}$$

$$\times \mathcal{R}\ell \iiint B_0(\underset{\sim}{u}_1) \; B_1(\underset{\sim}{u}_2) \; B_2(\underset{\sim}{u}_3) \; \mathcal{G}(\underset{\sim}{u}_2 - \underset{\sim}{u}_1)$$

$$\times \; \mathcal{G}(\underset{\sim}{u}_3 - \underset{\sim}{u}_2) \; \mathcal{G}(\underset{\sim}{u}_1 - \underset{\sim}{u}_3) \; d^2\underset{\sim}{u}_1 d^2\underset{\sim}{u}_2 d^2\underset{\sim}{u}_3, \tag{16}$$

where for brevity we have put

$$B(\underset{\sim}{u};\, \theta) = B_0(\underset{\sim}{u}), \qquad B(\underset{\sim}{u};\, \theta_i) = B_i(\underset{\sim}{u}), \qquad i = 1,\, 2, \tag{17}$$

and where[6]

$$\mathcal{G}(\underset{\sim}{u}) = A^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_A(\underset{\sim}{r}) \; \exp(ik\underset{\sim}{u}\cdot\underset{\sim}{r}/R) \; d^2\underset{\sim}{r} \tag{18}$$

is the Fourier transform of the aperture function $I_A(\underset{\sim}{r})$. Written in the same terms as Eq. (16), the threshold approximation of the ambiguity function is

$$H_0(\theta_1,\, \theta_2) = (E_S/N)^2 \; (TW)^{-1} \; B_T^{-2}$$

$$\times \iint B_1(\underset{\sim}{u}_1) \; B_2(\underset{\sim}{u}_2) \; |\mathcal{G}(\underset{\sim}{u}_1 - \underset{\sim}{u}_2)|^2 \; d^2\underset{\sim}{u}_1 d^2\underset{\sim}{u}_2. \tag{19}$$

For an ordinary aperture, $I_A(\underset{\sim}{r}) = 1$ for points $\underset{\sim}{r}$ within it, and $I_A(\underset{\sim}{r}) = 0$ for points outside, and its area A is

$$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_A(\underset{\sim}{r}) \; d^2\underset{\sim}{r}. \tag{20}$$

6. See ref. 1, Eq. (A4).

In order to assess the magnitude of $H_1(\theta_1, \theta_2)$, however, it is simpler to assume a Gaussian aperture,

$$I_A(\underset{\sim}{r}) = \exp\ (-\underset{\sim}{r}^2/2a^2) \tag{21}$$

of effective radius a and effective area

$$A = 2\ \pi\ a^2, \tag{22}$$

and to suppose that the object has a Gaussian distribution of radiance

$$B_0(\underset{\sim}{u}) = (B_T/A_o)\ \exp(-\underset{\sim}{u}^2/2b^2) \tag{23}$$

of effective radius b and effective area

$$A_o = 2\ \pi\ b^2. \tag{24}$$

The effective number M of spatial degrees of freedom in the aperture field is then[7]

$$M = \mathcal{F}^{-2} = 1 + 4b^2/\delta^2 = 1 + 4AA_o/\lambda^2R^2, \tag{25}$$

where

$$\delta = R/ka = \lambda R/2\pi a \tag{26}$$

is the radius of the function $\mathcal{G}(\underset{\sim}{u})$ in Eq. (18),

$$\mathcal{G}(\underset{\sim}{u}) = \exp(-\underset{\sim}{u}^2/2\delta^2). \tag{27}$$

We can now evaluate one of the integrals in (16),

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7. We use Eq. (3.18) of ref. 2 and Eqs. (3.8) and (A8) of ref. 1 to obtain this formula.

$$\int B_0(\underset{\sim}{u}_1)\ \mathcal{G}(\underset{\sim}{u}_2 - \underset{\sim}{u}_1)\ \mathcal{G}(\underset{\sim}{u}_1 - \underset{\sim}{u}_3)\ d^2\underset{\sim}{u}_1 =$$

$$(B_T/A_o)\int \exp\left[-\frac{\underset{\sim}{u}_1{}^2}{2b^2} - \frac{(\underset{\sim}{u}_2 - \underset{\sim}{u}_1)^2 + (\underset{\sim}{u}_1 - \underset{\sim}{u}_3)^2}{2\delta^2}\right] d^2\underset{\sim}{u}_1 =$$

$$B_T\ \delta^2(\delta^2 + 2b^2)^{-1} \exp\left[-\frac{(\underset{\sim}{u}_2 - \underset{\sim}{u}_3)^2}{4\delta^2} - \frac{(\underset{\sim}{u}_2 + \underset{\sim}{u}_3)^2}{4(\delta^2 + 2b^2)}\right]$$

$$= B_T\ \mathcal{G}'(\underset{\sim}{u}_2 - \underset{\sim}{u}_3)\ \ F(\underset{\sim}{u}_2 + \underset{\sim}{u}_3) \tag{28}$$

where

$$\mathcal{G}'(\underset{\sim}{u}) = \exp(-\underset{\sim}{u}^2/4\delta^2) \tag{29}$$

is nearly the same as $\mathcal{G}(\underset{\sim}{u})$ in (27), except that its radius is $\delta\sqrt{2}$ instead of δ, and where

$$F(\underset{\sim}{u}) = \delta^2(\delta^2 + 2b^2)^{-1} \exp[-\underset{\sim}{u}^2/4(\delta^2 + 2b^2)] \tag{30}$$

is at most equal to 1, but more generally

$$|F(\underset{\sim}{u})| \cong 2/(M + 1), \quad M \geq 1, \tag{31}$$

for most values of $\underset{\sim}{u}$. Putting these results into Eq. (16), we obtain, by comparison with Eq. (19),

$$H_1(\theta_1, \theta_2) \cong 2(E_S/N)^3\ (TW_3)^{-2}\ B_T{}^{-2}$$

$$R\ell \iint B_1(\underset{\sim}{u}_2)\ B_2(\underset{\sim}{u}_3)\ \mathcal{G}(\underset{\sim}{u}_3 - \underset{\sim}{u}_2)\ \mathcal{G}'(\underset{\sim}{u}_2 - \underset{\sim}{u}_3)$$

$$\times\ F(\underset{\sim}{u}_2 + \underset{\sim}{u}_3)\ d^2\underset{\sim}{u}_2 d^2\underset{\sim}{u}_3$$

$$\cong 4(E_S/N)\ (W/TW_3{}^2)\ (M + 1)^{-1}\ H_0(\theta_1, \theta_2). \tag{32}$$

We see from this that the threshold approximation $H_0(\theta_1, \theta_2)$ will be valid for the ambiguity function whenever

$$(E_s/N)\ [W/TW_3{}^2(M + 1)] << 1,$$

and since W_3 is of the order of W for most spectra, we require only that

$$E_s/TWN(M + 1) << 1,$$

which holds for objects of moderate detectability, for which the signal-to-noise ratio $E_s/N[(M + 1)\ TW]^{1/2}$ is of the order of 1.

II. Probability of Error in a Coherent Binary Optical Communication System

In an ideal quantum receiver of a coherent optical signal that excites only a single mode of the field, the optimum processor of the available data measures on the mode field a projection operator Π that is given by[8]

$$\Pi = \sum_k U(\eta_k) \, |\eta_k\rangle \langle\eta_k| \tag{1}$$

in terms of the eigenvectors $|\eta_k\rangle$ and the eigenvalues η_k of $\rho_1 - \lambda\rho_0$,

$$(\rho_1 - \lambda\rho_0) \, |\eta_k\rangle = \eta_k|\eta_k\rangle \, , \tag{2}$$

where ρ_0 is the density operator of the field when no signal is present, ρ_1 is the density operator when the signal is present, and

$$\lambda = \zeta/(1 - \zeta), \tag{3}$$

ζ being the prior probability that the signal is absent. In Eq. (1), $U(x)$ is the unit step function

$$\begin{aligned} U(x) &= 0, \; x < 0, \\ U(x) &= 1, \; x > 0. \end{aligned} \tag{4}$$

The average probability of error,

$$P_e = (1 - \zeta) \; [1 - \sum_k \eta_k \, U(\eta_k)] \tag{5}$$

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8. C. W. Helstrom, "Fundamental Limitations on the Detectability of Electromagnetic Signals", Int. J. Theoret. Phys. 1, 37-50, (May, 1968).

is the smallest attainable by any receiver processing the same data.

For a coherent signal in thermal noise, the density operators in the P-representation are

$$\rho_0 = (\pi\mathcal{N})^{-1} \int \exp(- |\alpha|^2/\mathcal{N}) \; |\alpha\rangle \langle\alpha| \; d^2\alpha, \tag{6}$$

$$\rho_1 = (\pi\mathcal{N})^{-1} \int \exp(- |\alpha - \mu|^2/\mathcal{N}) \; |\alpha\rangle \langle\alpha| \; d^2\alpha, \tag{7}$$

where

$$\mathcal{N} = [\exp(h\nu/K\mathcal{T}) - 1]^{-1} \tag{8}$$

is the average number of thermal photons per mode, h being Planck's constant, ν the frequency of the signal, K Boltzmann's constant, and $\mathcal{T}$ the effective absolute temperature of the noise. In (7), μ is the complex amplitude of the signal, so normalized that $N_s = |\mu|^2$ is the average number of photons contributed by the signal.

The probability of error P_e has been calculated analytically only for $\mathcal{N} = 0$ (no thermal noise) and for $\mathcal{N} \gg 1$ (the classical limit).[9] A numerical computation has now been made to evaluate P_e for intermediate values of $\mathcal{N}$, and for this it was assumed that $\zeta = 1 - \zeta = 1/2$ as in a binary communication system transmitting 0's and 1's with equal relative frequencies. For $\zeta = 1/2$, $\lambda = 1$,

$$P_e = \frac{1}{2} \{1 - [1 - \exp(-N_s)]^{1/2}\}, \; \mathcal{N} = 0, \tag{9}$$

and

$$P_e = \mathrm{erfc}\;(D/2), \; \mathcal{N} \gg 1, \tag{10}$$

are the analytical results in the two limiting cases, with

9. C. W. Helstrom, "Fundamentals Limitations on the Detectability of Electromagnetic Signals", Int. J. Theoret. Phys. 1, 37-50 (May, 1968).

$$D^2 = 4N_s/(2\mathcal{N} + 1) \tag{11}$$

the signal-to-noise ratio and

$$\operatorname{erfc} x = (2\pi)^{-1/2} \int_x^\infty \exp(-t^2/2)\,dt \tag{12}$$

the error-function integral.[10] These have been plotted in Fig. 1. In the classical limit, the signal-to-noise ratio becomes

$$D^2 \doteq 2E_s/K\mathcal{T} \tag{13}$$

where $E_s = N_s\, h\nu$ is the energy of the coherent signal.

For intermediate values of $\mathcal{N}$ the eigenvalue equation (2) was written in the number representation,

$$\sum_m (\rho_{1,nm} - \lambda\rho_{0,nm})\, x_m^{(k)} = \eta_k\, x_n^{(k)}, \tag{14}$$

$$\rho_{i,nm} = \langle n|\rho_i|m\rangle, \quad i = 0, 1,$$

$$x_m^{(k)} = \langle m|\eta_k\rangle, \tag{15}$$

where $|m\rangle$ are the eigenvectors of the number operator for the harmonic oscillator. The matrix elements $\rho_{1,nm}$ are[11]

$$\rho_{1,nm} = (1 - v)\ (n!/m!)^{1/2}\ (\mu^*/N)^{m-n}\, v^m$$

$$\times \exp[-(1 - v)\ |\mu|^2]\ L_n^{m-n}\ [-(1 - v)^2\ |\mu|^2/v],$$

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10. C. W. Helstrom, "Fundamental Limitations on the Detectability of Electromagnetic Signals", Int. J. Theoret. Phys. 1, 37-50 (May, 1968).

11. This formula was derived by applying Kummer's transformation to an expression derived by R. Yoshitani (UCLA thesis, unpublished).

$$v = \mathcal{N}/(\mathcal{N} + 1), \quad |\mu|^2 = N_s, \; m > n,$$

$$\rho_{1,nm} = \rho_{1,mn}{}^{*}, \; m < n, \tag{16}$$

where $L_n^{m-n}(x)$ is the associated Laguerre polynomial. The matrix $\rho_{0,nm}$ is diagonal,

$$\rho_{0,nm} = (1 - v) v^{m} \delta_{nm}. \tag{17}$$

It was then necessary to diagonalize the matrix $\rho_{1,nm} - \lambda\rho_{0,nm}$, $\lambda = 1$, and this was done by a digital computer working on the first M rows and columns of the matrix. For most cases M = 10 sufficed, although for D of the order of 3, M = 15 was necessary.

The resulting error probabilities P_e are plotted for $\mathcal{N} = 0.1$ and $\mathcal{N} = 0.2$ in Fig. 1. In Fig. 2 we exhibit the probability of error as a function of $\mathcal{N}$ for $N_s = 1$; this is the curve marked "quantum". The curve marked "classical" is a plot of Eq. (10) and represents a detector that measures the oscillator coordinate parallel to the signal amplitude μ. The optimum quantum detector provides a significantly lower probability of error only for $\mathcal{N} < 1$ when $N_s = 1$.

III. Discrete-Fourier-Transform Filters

(prepared by C. Rino)

In the previous report[12] we discussed the application of discrete Fourier transform methods to the approximation of least-squares filters. The approximation of the non-causal filter I-(6) was discussed. In this report we shall discuss the approximation of the causal filter I-(13).

To compute the filter I-(13) we must decompose certain hermitian functions into their analytic parts, i.e.

$$H^{+}(\omega) = \sum_{k=0}^{\infty} \alpha_k^{+} e^{-i\omega k} \tag{1}$$

and

$$H^{-}(\omega) = \sum_{k=0}^{\infty} \alpha_k^{-} e^{i\omega k} \tag{2}$$

in such a way that $H(\omega) = H^{+}(\omega) + H^{-}(\omega)$. Hence we have

$$\alpha_k^{+} = \begin{cases} \alpha_k, & k > 0, \\ \alpha_0/2, & k = 0, \\ 0, & k < 0, \end{cases} \tag{3}$$

and

$$\alpha_k^{-} = \begin{cases} 0, & k > 0, \\ \alpha_0/2, & k = 0, \\ \alpha_k, & k < 0, \end{cases} \tag{4}$$

where α_k is the k-th Fourier coefficient of $H(\omega)$. For the sake of computational

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12. "The Application of Signal Detection Theory to Optics", Quarterly Progress Report, NASA Grant NGR 05-009-079, September 15, 1969; See part 2, p. 4, hereafter referred to as I.

efficiency, we make use of only N uniformly spaced samples of $H(\omega)$ to compute $H^{+}(\omega)$. This immediately imposes a fundamental limitation on the accuracy with which we can compute α_k.[13] Hence we shall investigate a natural approximation to (1) and investigate its absolute error.

We consider the approximation

$$\tilde{H}^{+}\left(\frac{2\pi n}{N}\right) = \frac{1}{2}\alpha_{p_0} + \sum_{k=1}^{N/2} \alpha_{p_k} e^{-2\pi ink/N} + \frac{1}{2}\alpha_{p_{N/2}}(-1)^n, \tag{5}$$

where

$$\alpha_{p_k} = \sum_{m=-\infty}^{\infty} \alpha_{k+mN} = \frac{1}{N}\sum_{n=0}^{N-1} H\left(\frac{2\pi n}{N}\right) e^{2\pi ink/N}. \tag{6}$$

This approximation achieves the absolute error

$$E_N^{+}(n) = \tilde{H}^{+}\left(\frac{2\pi n}{N}\right) - H^{+}\left(\frac{2\pi n}{N}\right) =$$

$$\sum_{k=N/2+1}^{N-1} \alpha_{p_k}^{+} e^{-2\pi ink/N} - \sum_{k=1}^{N/2-1} \alpha_{p_k}^{-} e^{-2\pi ink/N}. \tag{7}$$

With some manipulation we can show that

$$E_N^{+}(n) = \sum_{m=0}^{\infty} (-i)^n \int_{-\pi}^{\pi} H\left(\omega + \frac{2\pi n}{N}\right) \frac{\sin\lambda\omega}{\sin\frac{1}{2}\omega} e^{i(m+3/4)N\omega} \frac{d\omega}{2\pi}$$

$$- i^n \int_{-\pi}^{\pi} H\left(\omega + \frac{2\pi n}{N}\right) \frac{\sin\lambda\omega}{\sin\frac{1}{2}\omega} e^{-i(m+3/4)N\omega} \frac{d\omega}{2\pi}, \tag{8}$$

where $\lambda = (N - 2)/4$. A refined analysis will show that the Fourier integrals in

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13. J. W. Tukey, "The Estimation of (Power) Spectra and Related Quantities", pp. 389-411, On Numerical Approximation, R. E. Langer Ed., University of Wisconsin Press, Madison, 1959.

(8) are asymptotically $O(N^{-K})$ where $H(\omega)$ has K continuous derivatives. It follows that the error $E_N^+(n)$ is asymptotically $O(N^{-K})$.

We first apply (5) to the canonical factorization of $\phi_{zz}(\omega)$. It can be shown that

$$\phi_{zz}^+(\omega) = \exp[\,\frac{1}{2}\log \phi_{zz}(\omega)]^+. \tag{9}$$

Hence we use the approximation

$$\tilde{\phi}_{zz}^+\left(\frac{2\pi n}{N}\right) = e^{g^+(n)}, \tag{10}$$

where $g^+(n)$ is the discrete-Fourier-transform approximation to the analytic part of $\frac{1}{2}\log \phi_{zz}(\omega)$. The error is purely imaginary and arbitrarily small for large N.

We mention here that these results can be applied to the continuous case as well as the discrete by making use of an isomorphic transformation described by Steiglitz.[14] After the application of the transformation, the Fourier coefficients become the Laguerre coefficients for the continuous-process spectrum. As an example we factored the spectrum

$$\varphi(\omega) = (1 + \omega^2/N)^{-N} \tag{11}$$

The approximation F(x) to the Fourier transform of $\varphi^+(\omega)$ is shown in Fig. 3 along with the absolute error. We used 128 points in the computation.

Now with the factorization complete we can again use (5) to approximate I-(14) and finally the causal filter I-(13). The mean-square error this filter achieves is currently being evaluated.

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14. K. Steiglitz, "The Equivalence of Digital and Analogue Signal Processing", Info. & Control, 8, 455-467 (October 1965).

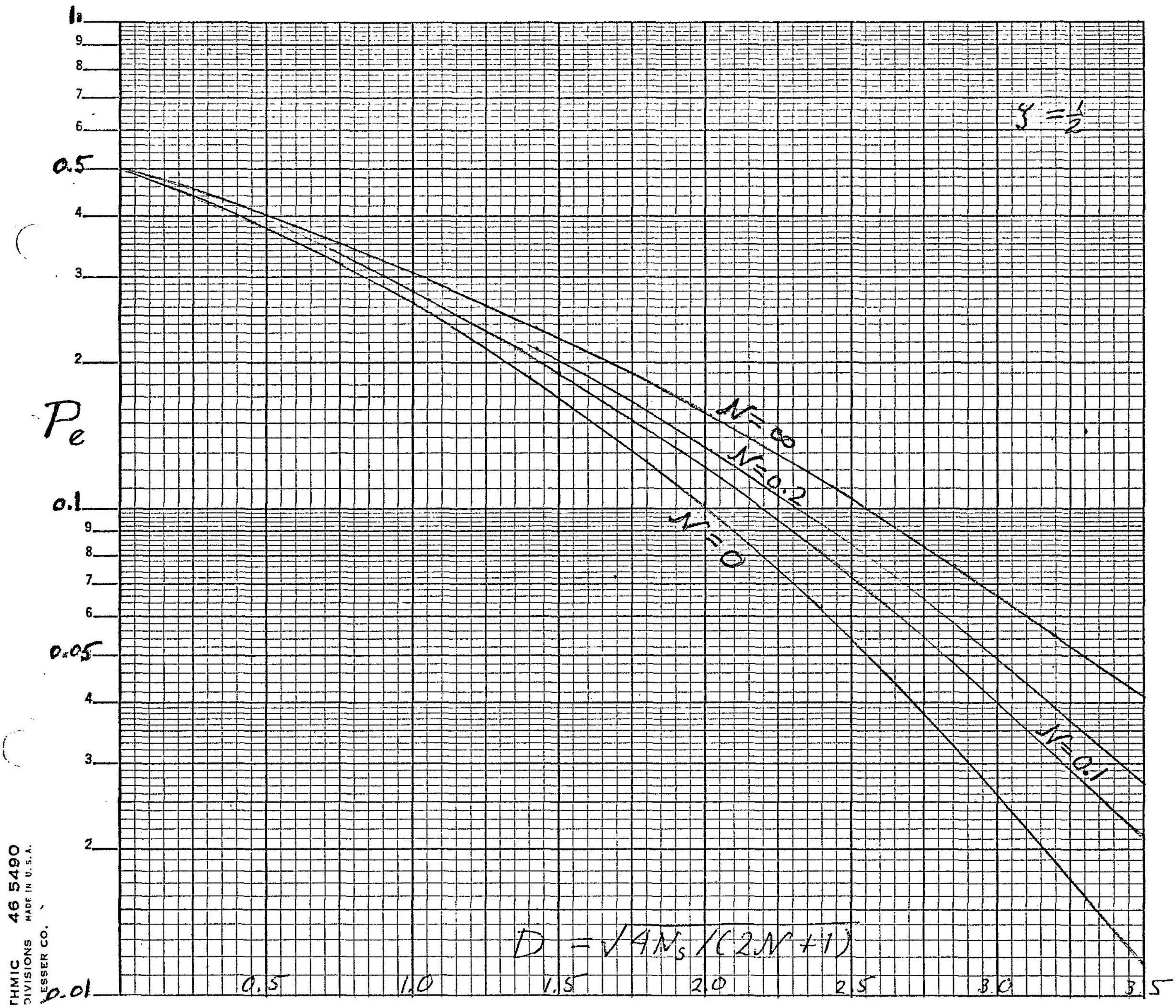

Fig. 1. Probability P_e of error in detection of known signal with prior probability $\zeta = 1/2$. D = signal-to-noise ratio = $[4N_s/(2\mathcal{N} + 1)]^{1/2}$, N_s = average number of signal photons, $\mathcal{N}$ = average number of thermal photons.

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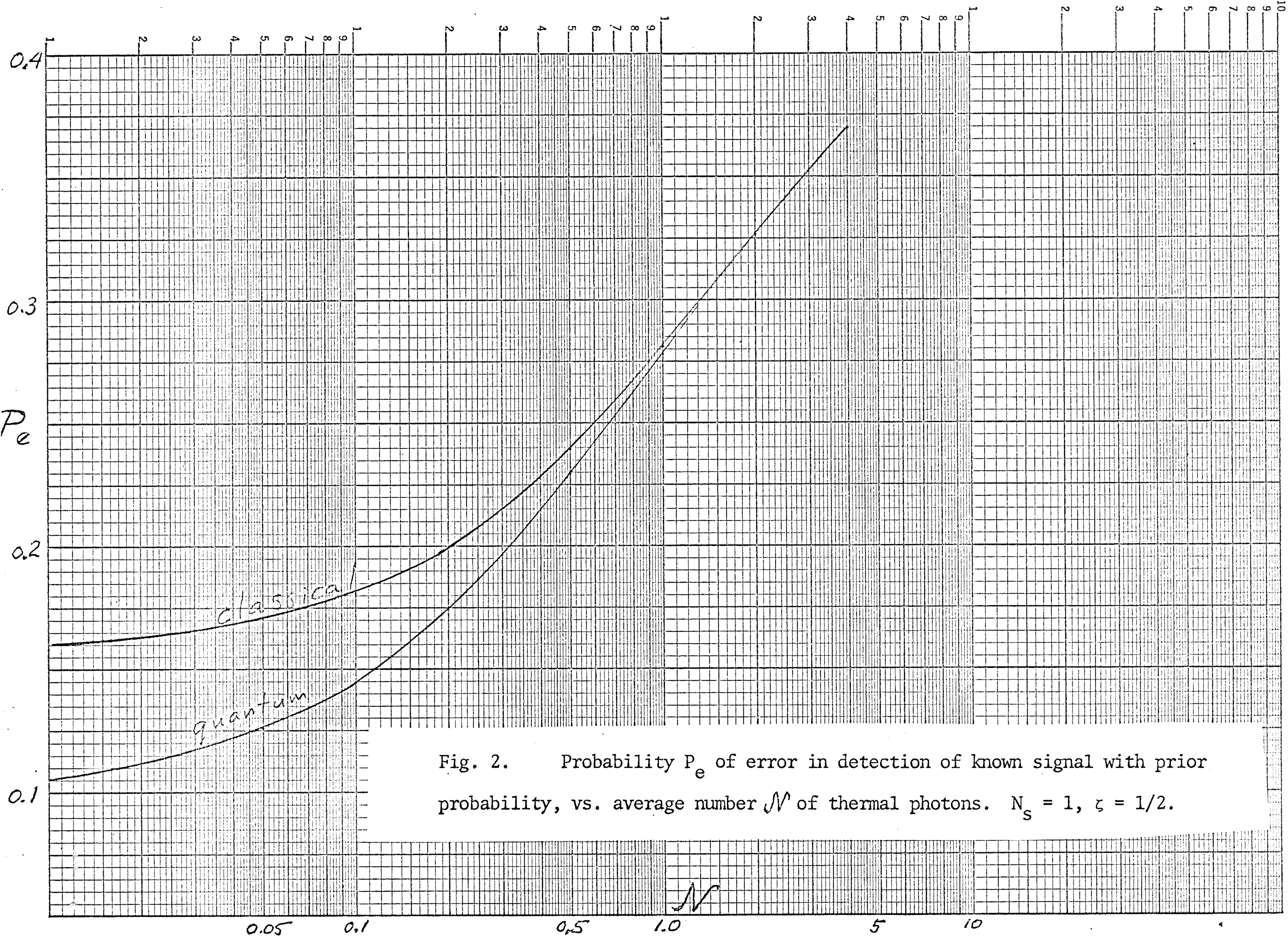

Fig. 2. Probability P_e of error in detection of known signal with prior probability, vs. average number $\mathcal{N}$ of thermal photons. $N_s = 1$, $\zeta = 1/2$.

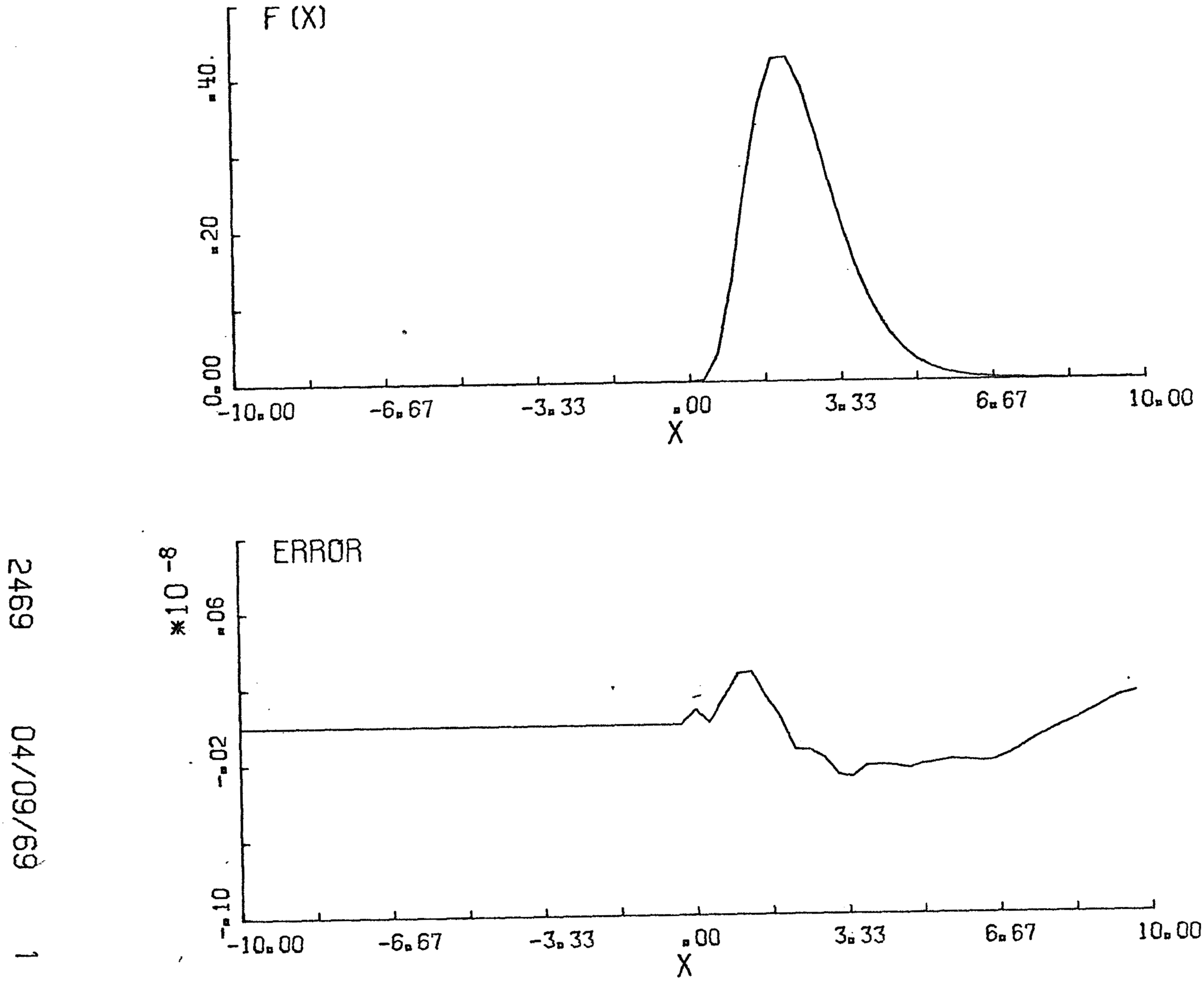

Fig. 3. Fourier transform of $\varphi^{+}(\omega)$ for the spectrum $\varphi(\omega) = (1 + \omega^2/N)^{-N}$, and its absolute error. Computation performed with 128 points and N = 6.